CLASSICAL MECHANICS

Three Hours

Do all fours problems

One Text Allowed

1. Consider a particle of mass 1 moving in a Newtonian (or Coulomb) potential $V(r) = -k/r$, where k is a "coupling constant" characterizing the attractive potential

   a) Show that the following vector (Runge-Lenz vector) is a constant of the motion:

   $$\vec{C} = \vec{v} \times \vec{A} - k\vec{r}/r,$$

   where $\vec{r}$ is the position vector, $\vec{v}$ is the velocity, and $\vec{A} = \vec{r} \times \vec{v}$ is the "vector area", constant by one of Kepler's laws.

   b) By cross-multiplying the vectors $\vec{C}$ and $\vec{A}$ show that the velocity vector traces out a circle and describe its relation to the vectors $\vec{C}$ and $\vec{A} \times \vec{C}$.

   c) What are the implications of the conserved quantity $\vec{C}$ in classical and in quantum mechanics?

2. Show that the angular momentum vector $\vec{J}$ of a rigid body with a given ellipsoid of inertia rotating with angular velocity $\vec{\omega}$ has the direction of the normal to the surface of the ellipsoid of inertia at the point where the rotation axis $\vec{\omega}$ meets the surface.

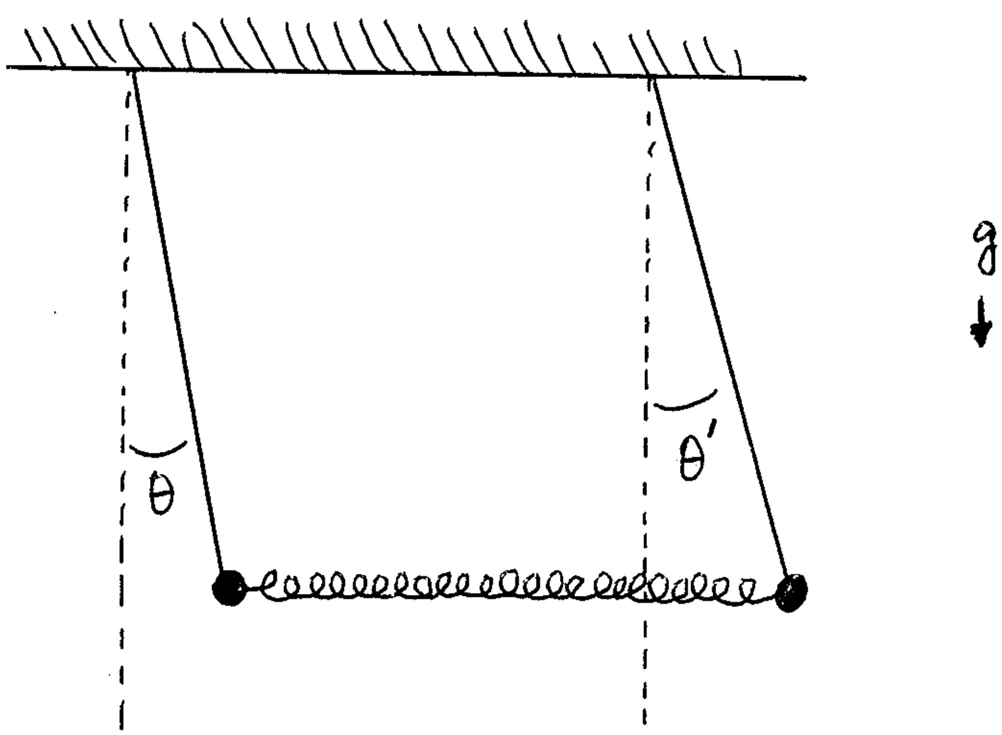
   Reminder: The ellipsoid of inertia is the locus of all angular momentum vectors of the body for which the kinetic energy equals 1/2; or the ellipsoid described by the moment-of-inertia tensor, having semi-axes inversely proportional to the square roots of the principal moments of inertia.

3. Consider a system consisting of two identical mathematical pendulums of length 1 and mass 1 placed in a uniform gravitational field with acceleration of gravity $g = 1$. The pendulums are connected by a weightless spring having equilibrium length equal to the distance between the points of suspension (see Figure), and spring constant k.

a) Choosing the angles θ and θ' as generalized coordinates write down the Lagrangian and the Hamiltonian of this system.

b) Considering the angles θ and θ' small (i.e., the angle equal to its sine) linearize the Lagrange equations by reducing the Lagrangian to the sum of two quadratic forms.

c) By simultaneous diagonalization of the corresponding matrices find the normal modes of the system, and for small values of k describe the vibrations qualitatively.



4. Use the Hamilton-Jacobi method to determine the motion of a symmetric two-dimensional harmonic oscillator:

   a) Write down the Lagrangian and Hamiltonian in cartesian and in polar coordinates.

   b) Write down the time-dependent Hamilton-Jacobi equation (in each of these coordinates systems).

   c) Find a complete integral of the H-J equation (e.g., by separation of variables, or utilization of cyclic variables).

   d) Use Jacobi's theorem to determine the trajectories in the plane and the motion along them.